Name: ______________________________

Lesson 3.7. Instantaneous Velocity

Date: __________________ Period: ________

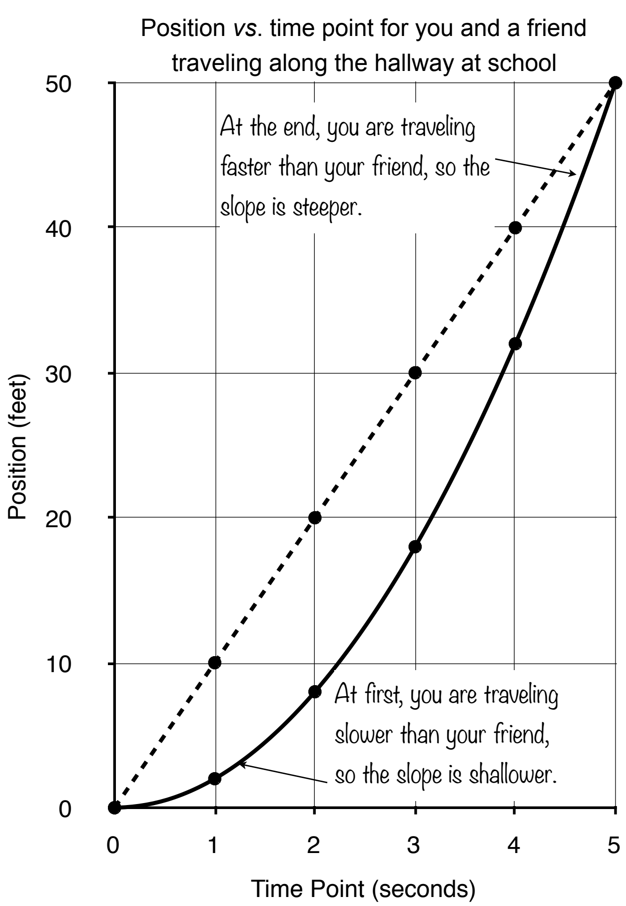
The Mean Value Theorem of Calculus

The distance and direction traveled *per unit time interval* is called the *velocity*. When an object travels equal distances during equal time intervals, the *velocity* is constant and we can easily determine its value. However, when an object travels different distances during equal time intervals, the velocity is *not* constant; in fact, in the case of the cardboard cylinder, the velocity is different at every instant of time. In such a case, how are we supposed to specify *a velocity* at all? It turns out that reporting a certain value of *velocity* only makes sense if you also specify the *time point* at which the velocity has that value. This is because at every other time point the velocity has a *different* value.

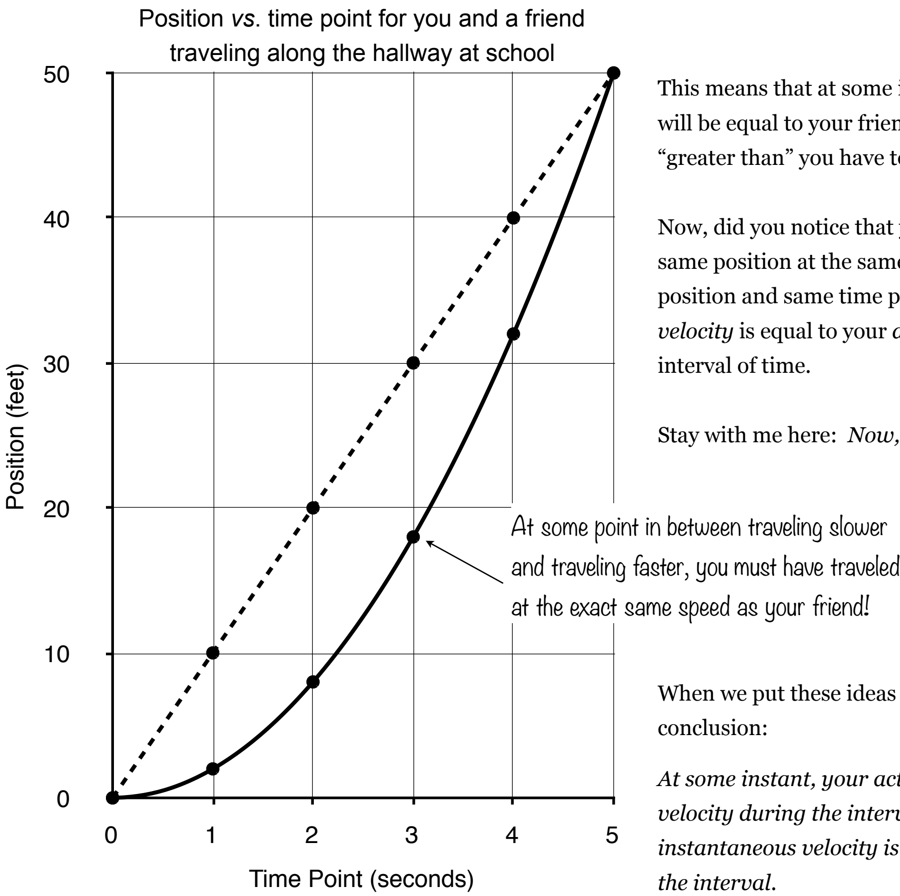
This quantity, *velocity at a specified time point*, is called the *instantaneous velocity* and its symbol is v_t. The t in the subscript stands for the specified time point. For example, the instantaneous velocity at $t = 5$ seconds is written v_5.

Perhaps you are wondering what *instantaneous velocity* actually means. Imagine you are sitting in your car at a stop light. When the light turns green, you accelerate from rest to, say, 45 mph. At some point, your speedometer needle will pass over the 30 mph reading; at that moment, your *instantaneous velocity* is 30 mph. How can you drive at 30 miles per hour for just an instant? Ah. That is a matter of rates and infinitesimals and you must study calculus to understand that. Or, just pay attention the next time you accelerate when a traffic light turns green!

In order to determine the value of *instantaneous velocity* at a time point of interest, one must use *calculus*, the branch of mathematics that analyzes continuous change. We shall invoke the *mean value theorem* from calculus to perform the calculations. Stay with me here, you can do this!



Begin by imagining that you are standing still and a friend races past you at a fast, constant speed. Starting from rest, you run faster and faster until you catch up. At first, you run more slowly than your friend because you started from rest. In order to catch up, you must increase your speed until you are running faster than your friend.



This means that at some instant, for just a moment, your velocity will be equal to your friend's. In order to go from "less than" to "greater than" you have to pass through "equal to."

Now, did you notice that you and your friend both began at the same position at the same time point, and ended at the same position and same time point? Thus, your friend's *constant velocity* is equal to your *average velocity* during that particular interval of time.

Stay with me here: *Now, concentrate!* I've just stated two ideas:

- ✦ At some instant, your velocity is equal to your friend's velocity;
- ✦ Your friend's velocity is equal to your average velocity.

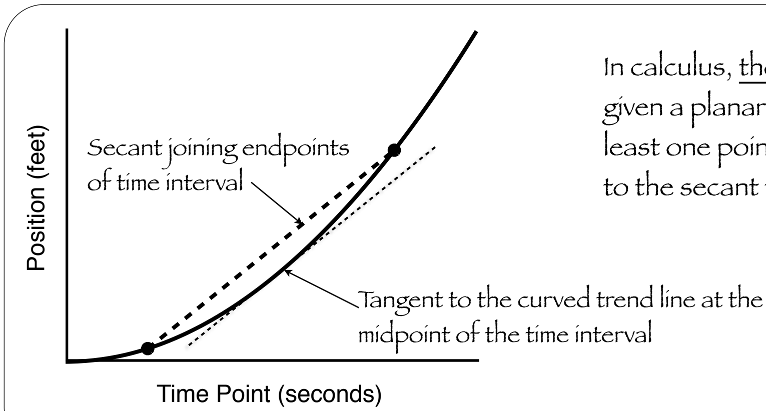
When we put these ideas together, we arrive at the following conclusion:

At some instant, your actual velocity is equal to your average velocity during the interval; that is, at some time point, your instantaneous velocity is equal to your average velocity during the interval.

Wait! It gets even better!

The Mean Value Theorem of calculus says that, ***for the special case of velocity that changes evenly***, the *average velocity over an interval* is equal to the *instantaneous velocity* ***at the midpoint in time of the interval!*** In the example above, this occurs at t = 3 s because that is the midpoint of the interval from t_i = 0 s to t_f = 5 s. At that moment, your velocity was equal to your friend's.

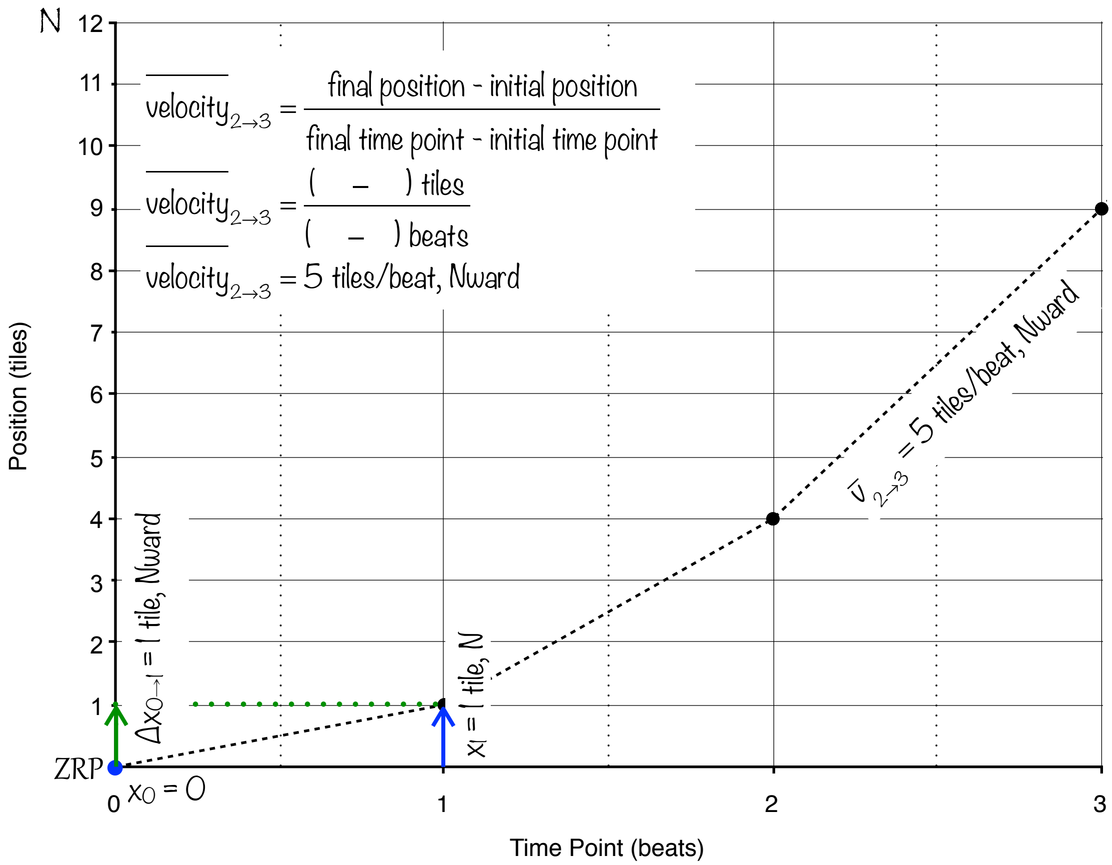
You already know how to determine whether the velocity changes evenly: if the displacement during equal time intervals increases or decreases evenly, then the *velocity* increases or decreases evenly. You also already know how to determine the *average velocity* during an interval. In short, you already know how to do everything you need in order to determine the instantaneous velocity at the midpoint of a time interval! How cool is that?



In calculus, the mean value theorem states, roughly: given a planar arc between two endpoints, there is at least one point at which the tangent to the arc is parallel to the secant through its endpoints. Just sayin'.[1]

[1] <http://en.wikipedia.org/wiki/Mean_value_theorem> March 14, 2013

We begin by modeling *evenly changing velocity* as a series of abrupt changes from one *constant velocity* to another *constant velocity*. On the position graph below, notice that I have drawn *straight*, dashed lines between the data points at 1 second intervals. Do you recognize these lines from your work on the previous lesson? These lines represent the *average velocity* over each time interval! So, we begin by determining the *average velocity* over each 1 second time interval.



First, you must determine whether the velocity changed evenly; that is, did the *displacement* change by equal amounts from one interval to the next?

- ❑ *Draw and label* a blue position arrow every 1 beat. If the position is the ZRP, then *draw and label* a blue dot.
- ❑ *Draw and label* a green *total displacement* arrow along with its dotted line at the beginning of each interval.
- ❑ Fill in the data table at right. Does the displacement change by equal amounts from one time interval to the next? Please carefully explain how you know.

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Does the displacement change evenly from one time interval to the next?

Time Interval	Displacement
0 beats → 1 beat	1 tile
1 beat → 2 beats	
2 beats → 3 beats	

Next, determine the *average velocity* of the cylinder during each time interval of 1 s. Review the previous lesson for help!

- ❑ Write *one* sample calculation right on the graph, if there's space. (I've left some blanks for you to fill in. *You're welcome.*)
- ❑ With your regular writing pencil, draw and label a straight, dashed line between t_i and t_f for each interval. (Oops, I've already drawn the dashed lines, so you must add only the labels. *You're welcome, again.*)

Now, according to the mean value theorem, *for the special case of evenly-changing velocity*, the *average velocity* over a time interval is equal to the *instantaneous velocity* at the midpoint of the interval.

☐ Carefully study the *average velocities* you recorded on the previous page, then fill in the empty cells in the data table below.

Time Interval	Average Velocity	Midpoint of Interval	Instantaneous Velocity
0 beats → 1 beat	1.0 tile/beat, Nward	0.5 beats	1.0 tile/beat, Nward
1 beat → 2 beats			
2 beats → 3 beats			

☐ Next, *draw and label* a red, *instantaneous velocity* arrow at each of these time points. (*I've already drawn one for you as an example. Please draw and label the other two red arrows.*)